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# On Asymptotic Expansion of the Conormal Symbol of the Singular Bochner-Martinelli Operator on the Surfaces with Singular Points

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*We study the conormal symbol of the singular Bochner-Martinelli integral on a compact closed surface with conical points  $\mathcal{S}$  in  $\mathbb{C}^n$  and evaluate its asymptotic expansion.*

*Key words:* singular Bochner-Martinelli operator, conormal symbol, conical point.

Smooth manifolds with conical points are the simplest singular spaces in the hierarchy of stratified varieties. Differential analysis on such manifolds was perhaps initiated by Kondrat'ev [1] who invented the so-called conormal symbol of a differential operator at a singular point.

In the 1980s the analysis encompassed also pseudodifferential operators which has led to diverse algebras of pseudodifferential operators on manifolds with conical points, see [4] and the references given there. All the algebras start actually with the same typical differential operators which are of Fuchs type.

When applied to the Cauchy integral on a plane curve with corners, conormal symbols can be efficiently computed.

The work [5] was intended as an attempt to examine whether the cone theory still effectively applies to higher dimensions. To this end, we study the singular Bochner-Martinelli integral on a compact closed surface with conical points  $\mathcal{S}$  in  $\mathbb{C}^n$  and evaluate its conormal symbol at a conical point. Our computation demonstrates rather strikingly that the conormal symbols are no longer efficient for pseudodifferential operators in dimensions larger than 1.

The singular Bochner-Martinelli integral is of central importance in complex analysis in several variables ([2]). In [3], the  $C^*$ -algebra generated by this integral on a compact closed hypersurface without singular points is described. In contrast to the singular case, the principal homogeneous symbol is as explicit as the Bochner-Martinelli integral itself.

In this work we find the asymptotic expansion of conormal symbol.

As usual, we identify  $\mathbb{C}^n$  with  $\mathbb{R}^{2n}$  under the complex structure  $z_j = x_j + ix_{n+j}$ , for  $j = 1, \dots, n$ .

We will consider a smooth hypersurface  $\mathcal{S}$  in  $\mathbb{C}^n \setminus \{0\}$  with a singular point at the origin given by

$$\mathcal{S} = \{(\varphi(r)x, r) \in \mathbb{R}^{2n} : x \in X, r \in [0, R)\}, \quad (1)$$

where  $\varphi \in C^1[0, R)$  satisfies  $\varphi(0) = 0$  and  $\varphi(r) > 0$  for  $r \in (0, R)$ , and the point  $x = (x_1, \dots, x_{2n-1})$  varies over a smooth compact hypersurface  $X$  in  $\mathbb{R}^{2n-1}$  which does not meet 0.

For instance,  $X$  may be a  $(2n-2)$ -dimensional sphere with the centre at the origin. In any case we assume that  $X = \{x \in \mathbb{R}^{2n-1} : \rho(x) = 1\}$ , where  $\rho$  is a  $C^1$  function on  $\mathbb{R}^{2n-1} \setminus \{0\}$  with real values, satisfying  $\nabla \rho \neq 0$  on  $X$  and  $\rho(\lambda x) = \lambda^h \rho(x)$  for all  $\lambda > 0$  with some  $h > 0$ .

The origin is a singular point of  $\mathcal{S}$ , for  $\varphi'(0) < \infty$ . If  $\varphi'(0) \neq 0$  then 0 is a conical point of  $\mathcal{S}$ . In the case  $\varphi'(0) = 0$  the point 0 is a cusp.

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Using (1) it is easy to determine a defining function of the smooth part of  $\mathcal{S}$ . Indeed, write  $\rho(x) = \rho(z', x_n)$ , where  $z' = (z_1, \dots, z_{n-1})$ . Then  $z \in \mathcal{S} \setminus \{0\}$  readily implies

$$\rho\left(\frac{z'}{\varphi(\operatorname{Im} z_n)}, \frac{\operatorname{Re} z_n}{\varphi(\operatorname{Im} z_n)}\right) = 1,$$

and so the homogeneity of  $\rho$  yields  $\mathcal{S} = \{z \in \mathbb{C}^n : \operatorname{Im} z_n \in [0, R), \varrho(z) = 0\}$ , with

$$\varrho(z) = \rho(z', \operatorname{Re} z_n) - (\varphi(\operatorname{Im} z_n))^h.$$

Given an integrable function  $f$  with compact support on  $\mathcal{S}$ , the singular Bochner-Martinelli integral of  $f$  is defined by

$$M_{\mathcal{S}} f(z) = \text{p.v.} \int_{\mathcal{S}} f(\zeta) U(\zeta, z)$$

for  $z \in \mathcal{S}$ , where

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n (-1)^{j-1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} d\bar{\zeta}[j] \wedge d\zeta$$

and  $d\zeta = d\zeta_1 \wedge \dots \wedge d\zeta_n$ , while  $d\bar{\zeta}[j]$  is the wedge product of all differentials  $d\bar{\zeta}_1, \dots, d\bar{\zeta}_n$  but  $d\bar{\zeta}_j$ . In the sequel, we drop the designation ‘p.v.’ for short.

The properties of the Bochner-Martinelli singular integral operator on the smooth hypersurfaces are well understood. We are aimed at investigating this operator on the hypersurfaces with isolated singular points. Since  $M_{\mathcal{S}} f$  is smooth away from the support of  $f$ , one can certainly assume without loss of generality that  $\mathcal{S}$  is of the form (1).

We first represent  $U(\zeta, z)$  in the local coordinates of  $\mathcal{S}$  close to a singular point. Set

$$\nu(y) = \frac{\nabla_y \rho}{|\nabla_y \rho|},$$

for  $y \in X$ , and  $\nu_{2n}(y, s) = -h \frac{\varphi'(s)}{|\nabla_y \rho|}$ .

**Lemma 1** ([5]). *The restriction of the Bochner-Martinelli kernel to the hypersurface  $\mathcal{S}$  has the form*

$$\begin{aligned} U(\zeta, z) &= \frac{1}{\sigma_{2n}} \frac{\langle (\nu(y), \nu_{2n}(y, s)), (\varphi(s)y - \varphi(r)x, s - r) \rangle}{(|\varphi(s)y - \varphi(r)x|^2 + (s - r)^2)^n} (\varphi(s))^{2n-2} ds d\sigma(y) - \\ &- i \frac{1}{\sigma_{2n}} \frac{\langle i\nu_c(y, s), (\varphi(s)y - \varphi(r)x, s - r) \rangle}{(|\varphi(s)y - \varphi(r)x|^2 + (s - r)^2)^n} (\varphi(s))^{2n-2} ds d\sigma(y), \end{aligned}$$

where  $d\sigma$  is the area form on  $X$  induced by the Lebesgue measure in  $\mathbb{R}^{2n}$ ,  $\sigma_{2n}$  the area of the  $(2n-1)$ -dimensional sphere, and  $i\nu_c = (-\nu_{n+1}, \dots, -\nu_{2n}, \nu_1, \dots, \nu_n)$ .

The vector  $i\nu_c$  is the vector lying in the tangent space to  $\mathcal{S}$  and such that it is orthogonal to complex tangent space to  $\mathcal{S}$ . It indicates to what extent the surface  $\mathcal{S}$  fits to the complex structure of  $\mathbb{C}^n$ , see [3].

From now on, we restrict our discussion to the hypersurfaces  $\mathcal{S} \subset \mathbb{R}^n$  with conical points.

Let  $\mathcal{D}$  be a bounded domain in  $\mathbb{C}^n$ , with  $n > 1$ . The boundary of  $\mathcal{D}$  is assumed to be of the form  $\mathcal{Y} \cup (\mathcal{S}_1 \cup \dots \cup \mathcal{S}_N)$ , where  $\mathcal{Y}$  is a smooth hypersurface and each  $\mathcal{S}_\nu$  is diffeomorphic to a conical hypersurface  $\mathcal{S}$  as above. Thus,  $\partial\mathcal{D}$  is a smooth hypersurface with a finite number of conical points. Since the analysis at singular points is local, one can assume without loss of generality that  $N = 1$ , i.e.,  $\partial\mathcal{D} = \mathcal{Y} \cup \mathcal{S}$  where

$$\mathcal{S} = \{z \in \mathbb{C}^n : z = (rx, r), x \in X, r \in [0, R)\},$$

cf. (1).

For the function  $f \in C_{\text{comp}}(\partial\mathcal{D} \setminus \{0\})$  we define the norm

$$\|f\|_{L^{2,\gamma}(\partial\mathcal{D})} := \left( \int_{\partial\mathcal{D}} |z|^{-2\gamma} |f|^2 d\Sigma \right)^{1/2}, \quad (2)$$

where  $\gamma \in \mathbb{R}$ . Denote by  $L^{2,\gamma}(\partial\mathcal{D})$  the completion of  $C_{\text{comp}}(\partial\mathcal{D} \setminus \{0\})$  with respect to this norm.

It is clear that the weight factor  $|z|^{-2\gamma}$  affects the behaviour of functions in  $L^{2,\gamma}(\partial\mathcal{D})$  merely at the conical point 0.

According to  $\partial\mathcal{D} = \mathcal{Y} \cup \mathcal{S}$ , the norm (2) can be splitted into two seminorms. The first of the two corresponds to integration over  $\partial\mathcal{D} \setminus \mathcal{S}$  and controls the behaviour of functions on the smooth part of the boundary. The second seminorm corresponds to integration over  $\mathcal{S}$  and specifies the behaviour of functions close to the singular points. Under the parametrisation (1), the hypersurface  $\mathcal{S}$  is identified with the cylinder  $X \times [0, R)$ . In this manner the second seminorm actually stems from the norm

$$\|f\|_{L^{2,\gamma-n+1/2}(X \times [0, R))} := \left( \int_0^R r^{-2(\gamma-n+1/2)} \|f\|_{L^2(X)}^2 \frac{dr}{r} \right)^{1/2}$$

on the functions  $f \in C_{\text{comp}}(X \times (0, R))$ .

Introduce a function  $k(x, y; t)$  defined for  $(x, y) \in X \times X$  and  $t > 0$  by the equality

$$\begin{aligned} k(x, y; t) &= \\ &= \frac{1}{\sigma_{2n}} \frac{\langle (\nu(y), \nu_{2n}(y)), (y - tx, 1 - t) \rangle}{(|y - tx|^2 + (1 - t)^2)^n} - \frac{i}{\sigma_{2n}} \frac{\langle i\nu_c(y), (y - tx, 1 - t) \rangle}{(|y - tx|^2 + (1 - t)^2)^n}. \end{aligned}$$

Using this kernel, we can rewrite the singular Bochner-Martinelli integral in the form

$$M_S f(x, r) = \int_0^\infty \frac{ds}{s} \int_X k\left(x, y; \frac{r}{s}\right) f(y, s) d\sigma(y), \quad (3)$$

where  $(x, r)$  and  $(y, s)$  are identified with  $z = (rx, r)$  and  $\zeta = (sy, s)$ , respectively. Note that the integral over  $X$  is singular, for  $k(x, y; r/s)$  has a singularity at  $y = x$  provided  $s = r$ .

**Theorem 1** ([5]). *Integral (3) induces a bounded linear operator in  $L^{2,\gamma}(X \times [0, R))$  provided  $1 - 2n < \gamma < 0$ .*

Denote by  $\mathcal{M}_{r \mapsto \lambda}$  the Mellin transform defined on functions  $f(r)$  on the semi-axis. It is given by

$$\mathcal{M}_{r \mapsto \lambda} f = \int_0^\infty r^{-i\lambda} f(r) \frac{dr}{r}$$

for  $\lambda \in \mathbb{C}$ .

Composing the singular Bochner-Martinelli operator (3) with the Mellin transform yields

$$\begin{aligned} \mathcal{M}_{r \mapsto \lambda} M_S f(x, r) &= \int_0^\infty r^{-i\lambda} \frac{dr}{r} \int_0^\infty \frac{ds}{s} \int_X k\left(x, y; \frac{r}{s}\right) f(y, s) d\sigma(y) = \\ &= \int_0^\infty \frac{ds}{s} \int_X \left( \int_0^\infty r^{-i\lambda} k\left(x, y; \frac{r}{s}\right) \frac{dr}{r} \right) f(y, s) d\sigma(y) \end{aligned}$$

where  $(x, r)$  and  $(y, s)$  are identified with the points  $z = (rx, r)$  and  $\zeta = (sy, s)$  of  $\mathcal{S}$ , respectively.

In the integral over  $r \in (0, \infty)$  we change the variables by  $r = st$ , where  $t$  runs over  $(0, \infty)$ . This gives

$$\begin{aligned} \mathcal{M}_{r \mapsto \lambda} M_S f(x, r) &= \int_0^\infty s^{-i\lambda} \frac{ds}{s} \int_X \left( \int_0^\infty t^{-i\lambda} k(x, y; t) \frac{dt}{t} \right) f(y, s) d\sigma(y) = \\ &= \int_X \mathcal{M}_{t \mapsto \lambda} k(x, y; t) \mathcal{M}_{s \mapsto \lambda} f(y, s) d\sigma(y) \end{aligned}$$

for  $x \in X$  and  $\lambda \in \mathbb{C}$ . It follows that

$$M_S f(r) = \mathcal{M}_{\lambda \mapsto r}^{-1} a(\lambda) \mathcal{M}_{r' \mapsto \lambda} f(r'), \quad (4)$$

where  $f(r) := f(x, r)$  is thought of as a function of  $r \in (0, \infty)$  with values in functions of  $x \in X$ , and  $a(\lambda)$  is a family of singular integral operators on  $X$  parametrised by  $\lambda$  varying on a horizontal line in the complex plane. The action of  $a(\lambda)$  is specified by

$$a(\lambda) f(x) = \int_X \mathcal{M}_{t \mapsto \lambda} k(x, y; t) f(y) d\sigma(y).$$

The family  $a(\lambda)$  is usually referred to as the conormal symbol of the pseudodifferential operator (3) based on the Mellin transform. To evaluate it more explicitly, we denote by  $Z$  the unique root of  $\langle y - tx, y - tx \rangle + (1 - t)^2 = 0$  in the upper half-plane, i.e.,

$$Z = \frac{1 + \langle x, y \rangle + i\sqrt{|y - x|^2 + |x|^2|y|^2 - \langle x, y \rangle^2}}{1 + |x|^2}. \quad (5)$$

**Lemma 2** ([5]). *In the strip  $0 < \text{Im } \lambda < 2n - 1$ , the Mellin transform of  $k(x, y; t)$  has the form*

$$\begin{aligned} \mathcal{M}_{t \mapsto \lambda} k(x, y; t) &= \\ &= \pi i \frac{(-1)^{n-1}}{(n-1)!} \frac{\exp \pi \lambda}{\sinh \pi \lambda} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!(n-1-j)!} (i\lambda+1)(i\lambda+2) \dots (i\lambda+j-1) \times \\ &\times \frac{((i\lambda+j)A - i\lambda ZB)Z^{-i\lambda-j-1} + (-1)^{j-1}((i\lambda+j)A - i\lambda \bar{Z}B)\bar{Z}^{-i\lambda-j-1}}{(1+|x|^2)^n (Z - \bar{Z})^{2n-1-j}}, \end{aligned}$$

where

$$\begin{aligned} A &= \frac{1}{\sigma_{2n}} \langle (\nu(y), \nu_{2n}(y)), (y, 1) \rangle - \frac{i}{\sigma_{2n}} \langle i\nu_c(y), (y, 1) \rangle, \\ B &= \frac{1}{\sigma_{2n}} \langle (\nu(y), \nu_{2n}(y)), (x, 1) \rangle - \frac{i}{\sigma_{2n}} \langle i\nu_c(y), (x, 1) \rangle. \end{aligned}$$

The Lemma is based on the formulas

$$\begin{aligned} \text{res}(f; Z) + \text{res}(f; \bar{Z}) &= \\ &= \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!(n-1-j)!} p(p-1) \dots (p-j+1) \frac{(-1)^{j+1} Z^{p-j} + \bar{Z}^{p-j}}{(Z - \bar{Z})^{2n-1-j}}, \end{aligned} \quad (6)$$

where

$$f(t) = \frac{t^p}{(t - Z)^n (t - \bar{Z})^n} \quad (7)$$

and

$$\int_0^\infty t^{-i\lambda} k(x, y; t) \frac{dt}{t} = \pi i \frac{\exp \pi \lambda}{\sinh \pi \lambda} \left( \text{res}(t^{-i\lambda-1} k(x, y; t); Z) + \text{res}(t^{-i\lambda-1} k(x, y; t); \bar{Z}) \right). \quad (8)$$

We denote

$$G(t) = t^{-\iota\lambda-1}k(x, y; t) = \frac{1}{a^n} \frac{At^{-\iota\lambda-1} - Bt^{-\iota\lambda}}{(t-Z)^n(t-\bar{Z})^n}.$$

We are now in a position to specify the inverse Mellin transform in the representation formula (4).

**Theorem 2** ([5]). *For  $|\gamma| < n - 1/2$  the singular Bochner-Martinelli integral admits the representation*

$$M_S f(r) = \frac{1}{2\pi} \int_{\operatorname{Im} \lambda = (n-1/2) - \gamma} r^{i\lambda} a(\lambda) \mathcal{M}_{r' \mapsto \lambda} f(r') d\lambda. \quad (9)$$

We first find asymptotics of the sum of residues of the function  $f(t)$  given by the formula (7).

**Lemma 3.** *The sum of residues of the function  $f(t)$  at  $Z$  and  $\bar{Z}$  has no singularity as  $\operatorname{Im} Z \rightarrow 0$ , and*

$$\lim_{\operatorname{Im} Z \rightarrow 0} \left( \operatorname{res}(f; Z) + \operatorname{res}(f; \bar{Z}) \right) = \frac{p(p-1) \dots (p-2n+2)}{(2n-1)!} Z^{p-2n+1}.$$

Moreover,

$$\begin{aligned} & \operatorname{res}(f; Z) + \operatorname{res}(f; \bar{Z}) = \\ &= \frac{1}{2(n-1)!} \sum_{s=0}^{\infty} \frac{p \dots (p-2n-s+2)(Z-\bar{Z})^s}{s!(s+n) \dots (s+2n-1)} \left( \bar{Z}^{p-2n-s+1} + (-1)^s Z^{p-2n-s+1} \right). \end{aligned}$$

PROOF. Set  $\Sigma = \operatorname{res}(f; Z) + \operatorname{res}(f; \bar{Z})$ . By the formula (6),

$$\begin{aligned} & \Sigma = \\ &= \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!(n-1-j)!} p(p-1) \dots (p-j+1) Z^{p-2n+1} \frac{(-1)^{j+1} + \left(\frac{\bar{Z}}{Z}\right)^{p-j}}{\left(1 - \frac{\bar{Z}}{Z}\right)^{2n-1-j}}. \end{aligned}$$

Setting  $Q := 1 - \bar{Z}/Z$  we rewrite  $\Sigma$  in the form

$$\frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!(n-1-j)!} p(p-1) \dots (p-j+1) Z^{p-2n+1} \frac{(-1)^{j+1} + (1-Q)^{p-j}}{Q^{2n-1-j}},$$

which splits into two sums

$$\begin{aligned} & Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!(n-1-j)!} p(p-1) \dots (p-j+1) \frac{(-1)^{j+1}}{Q^{2n-1-j}}, \\ & Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!(n-1-j)!} p(p-1) \dots (p-j+1) \frac{\sum_{k=0}^{\infty} \binom{p-j}{k} (-Q)^k}{Q^{2n-1-j}}. \end{aligned}$$

The binomial series in the latter sum converges only for  $|Q| < 1$ . If  $|Q| = 1$  it should be replaced by a Taylor polynomial of sufficiently large degree  $N$  along with a remainder  $O((\operatorname{Im} Z)^{N+1})$ .

Set  $l = j + k$  in the second sum and transform it. We obtain

$$Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \sum_{l=j}^{\infty} \frac{(2n-2-j)!}{j!(n-1-j)!} \frac{p(p-1) \dots (p-l+1)}{(l-j)!} \frac{(-1)^{l-j} Q^l}{Q^{2n-1}}.$$

Interchanging the order of summation and substituting  $j$  for  $l$  and  $k$  for  $j$  immediately yields

$$\begin{aligned} & Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \binom{p}{j} \frac{(-1)^j Q^j}{Q^{2n-1}} \sum_{k=0}^j (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} + \\ & + Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \sum_{j=n}^{\infty} \binom{p}{j} \frac{(-1)^j Q^j}{Q^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!}. \end{aligned}$$

Summarizing we get

$$\begin{aligned}
\Sigma &= Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!(n-1-j)!} p(p-1) \dots (p-j+1) \frac{(-1)^{j+1}}{Q^{2n-1-j}} + \\
&+ Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \binom{p}{j} \frac{(-1)^j Q^j}{Q^{2n-1}} \sum_{k=0}^j (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} + \\
&+ Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \sum_{j=n}^{\infty} \binom{p}{j} \frac{(-1)^j Q^j}{Q^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!}.
\end{aligned}$$

Lemma 3 will be proved once we prove the lemma below. The latter is of an independent interest.

**Lemma 4.** *We have*

$$\begin{aligned}
\sum_{k=0}^j (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} &= \frac{(2n-2-j)!}{(n-1-j)!}, \quad \text{if } j = 0, 1, \dots, n-1; \\
\sum_{k=0}^{n-1} (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} &= 0, \quad \text{if } j = n, n+1, \dots, 2n-2; \\
\sum_{k=0}^{n-1} (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} &= (-1)^{n-1} (n-1)!, \quad \text{if } j = 2n-1, \\
\sum_{k=0}^{n-1} (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} &= \\
= \sum_{l=2n-1}^j \frac{(-1)^{l+n} (l-2n+2) \dots (l-n)}{l!(j-l)!}, &\quad \text{if } j > 2n-1.
\end{aligned}$$

PROOF. Consider the function

$$F(z) = \sum_{k=0}^j (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} z^{n-1-k}.$$

A trivial verification shows that

$$\begin{aligned}
F(z) &= \left( \frac{\partial}{\partial z} \right)^{n-1} \sum_{k=0}^j (-1)^k \binom{j}{k} z^{2n-2-k} = \\
&= \left( \frac{\partial}{\partial z} \right)^{n-1} z^{2n-2} \left( 1 - \frac{1}{z} \right)^j = \\
&= \left( \frac{\partial}{\partial z} \right)^{n-1} (z^{2n-2-j} (z-1)^j).
\end{aligned}$$

For  $z = 1$  we then get

$$\begin{aligned}
\sum_{k=0}^j (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} &= \binom{n-1}{j} j! (2n-2-j)(2n-3-j) \dots n = \\
&= \frac{(2n-2-j)!}{(n-1-j)!}
\end{aligned}$$

whenever  $j = 0, 1, \dots, n-1$ .

In just the same way we evaluate the second sum. Suppose  $j = n, n+1, \dots, 2n-2$ . Consider the function

$$F(z) = \sum_{k=0}^{n-1} (-1)^k \binom{j}{k} \frac{(2n-2-k)!}{(n-1-k)!} z^{n-k-1}$$

which is actually equal to

$$\left(\frac{\partial}{\partial z}\right)^{n-1} (z^{2n-2-j}(z-1)^j),$$

as it is easy to check. Hence we readily deduce that  $F(1) = 0$ , as desired.

Let us prove the third equality corresponding to  $j = 2n - 1$ . For this purpose, consider the function

$$F(z) = \sum_{k=0}^{n-1} (-1)^k \binom{2n-1}{k} \frac{(2n-2-k)!}{(n-1-k)!} z^{n-k-1}.$$

An easy computation shows that

$$\begin{aligned} F(z) &= \left(\frac{\partial}{\partial z}\right)^{n-1} \sum_{k=0}^{n-1} (-1)^k \binom{2n-1}{k} z^{2n-2-k} = \\ &= \left(\frac{\partial}{\partial z}\right)^{n-1} \left( \sum_{k=0}^{2n-1} (-1)^k \binom{2n-1}{k} z^{2n-2-k} - (-1)^{2n-1} \frac{1}{z} \right) = \\ &= \left(\frac{\partial}{\partial z}\right)^{n-1} \left( \frac{(z-1)^{2n-1}}{z} \right) + \left(\frac{\partial}{\partial z}\right)^{n-1} \frac{1}{z} = \\ &= \left(\frac{\partial}{\partial z}\right)^{n-1} \left( \frac{(z-1)^{2n-1}}{z} \right) + (-1)^{n-1} (n-1)! z^{-n}. \end{aligned}$$

For  $z = 1$  the first term vanishes, and so  $F(1) = (-1)^{n-1} (n-1)!$

Consider the last equality corresponding to  $j > 2n - 1$ . We have

$$\begin{aligned} F(z) &= \left(\frac{\partial}{\partial z}\right)^{n-1} \left( \sum_{l=0}^{n-1} \frac{(-1)^l z^{2n-l-2}}{l!(j-l)!} \right) = \\ &= \left(\frac{\partial}{\partial z}\right)^{n-1} \left( \sum_{l=0}^j \frac{(-1)^l z^{2n-l-2}}{l!(j-l)!} - \sum_{l=2n-1}^j \frac{(-1)^l z^{2n-l-2}}{l!(j-l)!} \right) = \\ &= \left(\frac{\partial}{\partial z}\right)^{n-1} \left( \frac{z^{2n-j-2}(z-1)^j}{j!} \right) + \left(\frac{\partial}{\partial z}\right)^{n-1} \left( \sum_{l=2n-1}^j \frac{(-1)^{l+1} z^{2n-l-2}}{l!(j-l)!} \right). \end{aligned}$$

Therefore

$$F(1) = \sum_{2n-1}^j \frac{(-1)^{l+n}(l-2n+2) \cdots (l-n)}{l!(j-l)!},$$

which proves the lemma.  $\square$

We are now in a position to complete the proof of Lemma 3. To this end, we observe that the first and the second sums in the expression for  $\Sigma$  cancel. In the third sum only the terms corresponding to  $j \geq 2n - 1$  do not vanish. Hence it follows that

$$\begin{aligned} \lim_{\text{Im } Z \rightarrow 0} \Sigma &= Z^{p-2n+1} \frac{(-1)^n}{(n-1)!} \binom{p}{2n-1} \frac{(-1)^{2n-1} Q^{2n-1}}{Q^{2n-1}} = (-1)^{n-1} (n-1)! \\ &= \binom{p}{2n-1} Z^{p-2n+1}. \end{aligned}$$

Then by Lemma 4

$$\Sigma = Z^{p-2n+1} \sum_{j=2n-1}^{\infty} (-1)^j p(p-1) \cdots (p-j+1) Q^{j-2n-1} \sum_{l=2n-1}^j \frac{(-1)^{l+n}(l-2n+2) \cdots (l-n)}{l!(j-l)!}.$$

Substituting  $j = k + 2n - 1$  and  $l = s + 2n - 1$ , we get

$$\Sigma = Z^{p-2n+1} \sum_{k=0}^{\infty} (-1)^{k+1} p \cdots (p-k-2n+2) Q^k \sum_{s=0}^k \frac{(-1)^{s+n-1}(s+1) \cdots (s+n-1)}{(s+2n-1)!(k-s)!} =$$

$$\begin{aligned}
&= Z^{p-2n-1} \sum_{s=0}^{\infty} \sum_{k=s}^{\infty} \frac{(-1)^{k+s+n} p \cdots (p-k-2n+2)(s+1) \cdots (s+n-1) Q^k}{(s+2n-1)!(k-s)!} = \\
&= Z^{p-2n+1} \sum_{s=0}^{\infty} \frac{(-1)^{s+n} (j+1) \cdots (j+n-1)}{(s+2n-1)!} \sum_{k=s}^{\infty} \frac{(-1)^k p \cdots (p-k-2n+2) Q^k}{(k-s)!} = \\
&= Z^{p-2n+1} \sum_{s=0}^{\infty} \frac{(-1)^{s+n} (j+1) \cdots (j+n-1)}{(s+2n-1)!} \sum_{m=0}^{\infty} \frac{(-1)^{s+m} p \cdots (p-s-m-2n+2) Q^{s+m}}{m!} = \\
&= Z^{p-2n+1} \sum_{s=0}^{\infty} \frac{(-1)^n (j+1) \cdots (j+n-1) Q^s}{(s+2n-1)!} \sum_{m=0}^{\infty} \frac{(-1)^m p \cdots (p-s-m-2n+2) Q^m}{m!}.
\end{aligned}$$

The sum

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(-1)^m p \cdots (p-s-m-2n+2) Q^m}{m!} = \\
&= p \cdots (p-s-2n+2) \sum_{m=0}^{\infty} \frac{(-1)^m (p-2n-s+1) \cdots (p-s-m-2n+2) Q^m}{m!} = \\
&= p \cdots (p-s-2n+2) (1-Q)^{p-2n-s-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Sigma &= Z^{p-2n+1} \sum_{s=0}^{\infty} \frac{(-1)^n (s+1) \cdots (s+n-1) p \cdots (p-2n-s+2) Q^s (1-Q)^{p-2n-s+1}}{(s+2n-1)!} = \\
&= (-1)^n \cdot Z^{p-2n+1} \cdot (1-Q)^{p-2n+1} \sum_{s=0}^{\infty} \frac{(s+1) \cdots (s+n-1) p \cdots (p-2n-s+2) Q^s (1-Q)^{-s}}{(s+2n-1)!}.
\end{aligned}$$

Since  $Q = 1 - \bar{Z}/Z$ , then  $1 - Q = \bar{Z}/Z$  and

$$\frac{Q}{1-Q} = \frac{Z - \bar{Z}}{\bar{Z}}.$$

From here

$$\Sigma = (-1)^n \bar{Z}^{p-2n-1} \sum_{s=0}^{\infty} \frac{p \cdots (p-2n-s+2)}{s!(s+n) \cdots (s+2n-1)} \cdot \frac{(Z - \bar{Z})^s}{\bar{Z}^s}.$$

Hence

$$\begin{aligned}
\text{res}(f; Z) + \text{res}(f; \bar{Z}) &= \frac{\bar{Z}^{p-2n+1}}{(n-1)!} \sum_{s=0}^{\infty} \frac{p \cdots (p-2n-s+2)(Z - \bar{Z})^s}{s!(s+n) \cdots (s+2n-1) \bar{Z}^s} = \\
&= \frac{1}{2(n-1)!} \sum_{s=0}^{\infty} \frac{p \cdots (p-2n-s+2)(Z - \bar{Z})^s}{s!(s+n) \cdots (s+2n-1)} \cdot \left( \bar{Z}^{p-2n-s+1} + (-1)^s Z^{p-2n-s+1} \right),
\end{aligned}$$

as desired.  $\square$

**Theorem 3.** *The function  $\mathcal{M}_{t \mapsto \lambda} k(x, y; t)$  admits an asymptotic expansion*

$$\begin{aligned}
\mathcal{M}_{t \mapsto \lambda} k(x, y; t) &= \\
&= \pi \frac{(i\lambda + 1) \cdots (i\lambda + 2n - 2)}{(2n-1)!} \frac{\exp \pi \lambda ((i\lambda + 2n - 1) \text{Im } A - i\lambda \text{Re } Z \text{Im } B) Z^{-i\lambda-2n}}{\sinh \pi \lambda (1 + |x|^2)^n} + \\
&+ O(\text{Im } Z)
\end{aligned}$$

as  $\text{Im } Z \rightarrow 0$ .

Moreover,

$$\begin{aligned}
\mathcal{M}_{t \mapsto \lambda} k(x, y; t) &= \frac{i\pi}{2(2n-1)!} \frac{\exp \pi \lambda}{(1 + |x|^2)^n \sinh \pi \lambda} \sum_{s=0}^{\infty} \frac{(i\lambda + 1) \cdots (i\lambda + 2n + s - 2)(Z - \bar{Z})^s}{s!(s+n) \cdots (s+2n-1)} \times \\
&\times ((-1)^{s+1} \bar{Z}^{-i\lambda-2n-s} (i\lambda B \bar{Z} + A(i\lambda + 2n + s - 1)) - \bar{Z}^{-i\lambda-2n-s} (i\lambda B Z + A(i\lambda + 2n + s - 1))).
\end{aligned}$$



PROOF. Using Lemma 2 and Lemma 3 we obtain

$$\begin{aligned} \lim_{\operatorname{Im} Z \rightarrow 0} \mathcal{M}_{t \rightarrow \lambda} k(x, y; t) &= \\ &= -\pi i \frac{(i\lambda + 1) \cdots (i\lambda + 2n - 2)}{(2n - 1)!} \frac{\exp \pi \lambda}{\sinh \pi \lambda} \frac{((i\lambda + 2n - 1)A - i\lambda ZB)Z^{-i\lambda - 2n}}{(1 + |x|^2)^n}. \end{aligned}$$

Let us estimate the sum  $B + \bar{B}$ . Since  $\langle \nabla_y \rho, y \rangle = h$  it follows that the real part of  $B$  is

$$\frac{B + \bar{B}}{2} = \frac{2}{\sigma_{2n}} \frac{\langle \nabla_y \rho, x \rangle - h}{|\nabla_y \rho|} = \frac{2}{\sigma_{2n}} \frac{\langle \nabla_y \rho, x - y \rangle}{|\nabla_y \rho|},$$

which is  $O(\operatorname{Im} Z)$  as  $\operatorname{Im} Z \rightarrow 0$ . On the other hand,  $A$  is purely imaginary, for

$$\langle (\nu(y), \nu_{2n}(y)), (y, 1) \rangle = 0.$$

This establishes the first formula.

Consider the last formula. Since

$$\begin{aligned} &\operatorname{res}(G; Z) + \operatorname{res}(G; \bar{Z}) = \\ &= \frac{B}{2(n-1)!a^n} \sum_{s=0}^{\infty} \frac{(-i\lambda) \cdots (-i\lambda - 2n - s + 2)(Z - \bar{Z})^s}{s!(s+n) \cdots (s+2n-1)} (\bar{Z}^{-i\lambda - 2n - s + 1} + (-1)^s Z^{-i\lambda - 2n - s + 1}) + \\ &+ \frac{A}{2(n-1)!a^n} \sum_{s=0}^{\infty} \frac{(-i\lambda - 1) \cdots (-i\lambda - 2n - s + 1)(Z - \bar{Z})^s}{s!(s+n) \cdots (s+2n-1)} (\bar{Z}^{-i\lambda - 2n - s} + (-1)^s Z^{-i\lambda - 2n - s}) = \\ &= \frac{1}{2(n-1)!a^n} \sum_{s=0}^{\infty} \frac{(-i\lambda + 1) \cdots (-i\lambda - 2n - s - 2)(Z - \bar{Z})^s}{s!(s+n) \cdots (s+2n-1)} \times \\ &\times ((-1)^{s+1} (i\lambda B \bar{Z}^{-i\lambda - 2n - s + 1} + A(i\lambda + 2n + s - 1) \bar{Z}^{-i\lambda - 2n - s}) - \\ &- (i\lambda B Z^{-i\lambda - 2n - s + 1} + A(i\lambda + 2n + s - 1) Z^{-i\lambda - 2n - s})) = \\ &= \frac{1}{2(n-1)!a^n} \sum_{s=0}^{\infty} \frac{(-i\lambda + 1) \cdots (-i\lambda - 2n - s - 2)(Z - \bar{Z})^s}{s!(s+n) \cdots (s+2n-1)} \times \\ &\times ((-1)^{s+1} \bar{Z}^{-i\lambda - 2n - s} (\lambda B \bar{Z} + A(i\lambda + 2n + 3 - 1)) - \\ &- Z^{-i\lambda - 2n - s} (\lambda B Z + A(i\lambda + 2n + 3 - 1))). \end{aligned}$$

Then, using the equality (8), we get

$$\begin{aligned} &\int_0^\infty t^{-i\lambda} k(x, y; t) \frac{dt}{t} = \pi i \frac{\exp \pi \lambda}{\sinh \pi \lambda} \left( \operatorname{res}(G(t); Z) + \operatorname{res}(G(t); \bar{Z}) \right) = \\ &= \frac{i\pi \exp \pi \lambda}{2(n-1)!a^n \sinh \pi \lambda} \sum_{s=0}^{\infty} \frac{(-i\lambda + 1) \cdots (-i\lambda - 2n - s - 2)(Z - \bar{Z})^s}{s!(s+n) \cdots (s+2n-1)} \times \\ &\times ((-1)^{s+1} \bar{Z}^{-i\lambda - 2n - s} (\lambda B \bar{Z} + A(i\lambda + 2n + 3 - 1)) - \\ &- Z^{-i\lambda - 2n - s} (\lambda B Z + A(i\lambda + 2n + 3 - 1))). \end{aligned}$$

□

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